ALGEBRAIC CURVES
EXERCISE SHEET 2

Exercise 2.1.

(1) Let V be an algebraic set in $\mathbf{A}^n(k)$ and $P \in \mathbf{A}^n(k)$ a point not in V. Show that there is a polynomial $F \in k[X_1, ..., X_n]$ such that $F(Q) = 0$ for all $Q \in V$, but $F(P) = 1$.
(2) Let $P_1, ..., P_r$ be distinct points in $\mathbf{A}^n(k)$, not in an algebraic set V. Show that there are polynomials $F_1, ..., F_r \in I(V)$ such that $F_i(P_j) = 0$ if $i \neq j$, and $F_i(P_i) = 1$.
(3) With $P_1, ..., P_r$ and V as in (2),and $a_{ij} \in k$ for $1 \leq i, j \leq r$, show that there are $G_i \in I(V)$ with $G_i(P_j) = a_{ij}$ for all i and j.

Exercise 2.2.

(1) Determine which of the following sets are algebraic:
   (a) $\{(x, y) \in \mathbf{A}^2(\mathbf{R}) \mid y = sin(x)\}$
   (b) $\{(cos(t), sin(t)) \in \mathbf{A}^2(\mathbf{R}) \mid t \in \mathbf{R}\}$
   (c) $\{(z, w) \in \mathbf{A}^2(\mathbf{C}) \mid |z|^2 + |w|^2 = 1\}$
(2) Show that any algebraic subset of $\mathbf{A}^n(\mathbf{R})$ can be defined by a single polynomial equation. Is the same true for $\mathbf{A}^n(\mathbf{C})$?

Exercise 2.3.

Let k be a field and I, J two ideals of $k[x_1, \ldots, x_n]$. Let $a = (a_1, \ldots, a_n) \in k^n$. Recall that $I \cdot J = (\{fg,\ f \in I,\ g \in J\})$. Show the following assertions:

(1) If $I \subseteq J$, then $V(J) \subseteq V(I)$.
(2) $V(I) \cup V(J) = V(I \cdot J)$.
(3) $V(\{x_1 - a_1, \ldots, x_n - a_n\}) = \{a\}$.

Exercise 2.4.

Let $V \subseteq \mathbb{A}_k^n$ and $W \subseteq \mathbb{A}_k^m$ be algebraic sets. Show that the following set is an algebraic subset of $\mathbb{A}_k^{m+n}$:

$$V \times W = \{(a_1, \ldots, a_n, b_1, \ldots, b_m) \in \mathbb{A}_k^{m+n} \mid (a_1, \ldots, a_n) \in V,\ (b_1, \ldots, b_m) \in W\}$$

Exercise 2.5.

A ring is called *local* if it has a unique maximal ideal. Let k be an algebraically closed field.

(1) Let $I \subseteq R = k[x_1, \dots, x_n]$ be an ideal such that $V(I)$ is a point. Show that R/I is a finite-dimensional local algebra and that the elements of the maximal ideal are nilpotent.
(2) Let $I \subseteq R = k[x_1, \dots, x_n]$ be a *radical* ideal such that $V(I)$ is a finite set of r points. Show that $R/I \simeq k \times \dots \times k$, with r copies of k in the product. (Hint: consider the intersection of maximal ideals containing I and use the chinese remainder theorem).

Exercise 2.6.

Let k be an algebraically closed field and $V = \{p_1, \dots, p_r\} \subseteq \mathbb{A}_k^n$ a finite algebraic set. We call a_i, $1 \leq i \leq s$ the distinct first coordinates of $p_1, \dots, p_r$. Consider the finite varieties $V_i = \{(x_2, \dots, x_n) \in \mathbb{A}^{n-1} \mid (a_i, x_2, \dots, x_n) \in V\} \subseteq \mathbb{A}^{n-1}$.

(1) Assume that each V_i is the zero locus of N polynomials $f_{i,1}, \dots, f_{i,N}$ for some $N \geq 1$. Show that there exist polynomials g_k, $1 \leq k \leq N$ such that $g_k(a_i, x_2, \dots, x_n) = f_{i,k}$.
(2) Show that V is the zero locus of n polynomials in $k[x_1, \dots, x_n]$. (Hint: reason by induction on n)
(3) Show that $I(V)$ is generated by n polynomials. (Hint: using the previous exercise, $I(V)$ is characterized by $k[x_1, \dots, x_n]/I(V) \simeq k \times \dots \times k$)